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STURM–LIOUVILLE PROBLEM FOR A DIFFERENTIAL EQUATION OF SECOND ORDER WITH DISCONTINUOUS COEFFICIENTS

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Using, as an example, a special Sturm–Liouville boundary-value problem for a differential equation of second order with discontinuous coefficients, the authors describe a method of constructing a closed orthonormalized system of functions that is common to the entire domain of determination.

As is known [1-4], when the Fourier method is employed to solve some classical problems of mathematical physics, one must solve the following boundary-value problem for the eigenvalues λ_n and eigenfunctions $u_n(x)$:

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+\lambda^2 r(x)\,u=0\,, \tag{1}$$

$$\alpha_1 U(a)+\beta_1 U'(a)=0\,, \tag{2}$$

$$\alpha_2 U(b)+\beta_2 U'(b)=0\,. \tag{3}$$

In the interval $[a, b]$ the function $p(x)$ does not vanish and has a discontinuous derivative.

Let us consider the nonclassical case where the coefficients of Eq. (1) are discontinuous functions of the coordinate x. We arrive at a Sturm–Liouville problem of this kind when, for instance, boundary-value problems of unsteady heat conduction for piecewise-homogeneous bodies are solved by the Fourier method.

Let

$$p(x)=p_1+\sum_{i=1}^{n-1}(p_{i+1}-p_i)\,S_-(x-x_i)\,, \tag{4}$$

$$r(x)=r_1+\sum_{i=1}^{n-1}(r_{i+1}-r_i)\,S_-(x-x_i)\,, \tag{5}$$

i.e., the coefficients of differential equation (1) are step functions of the coordinate x. Here, $S_-(x-x_i)$ denotes the asymmetric unit function

$$S_-(x-x_i)=\begin{cases}0\,, & x<x_i\,,\\ 1\,, & x\ge x_i\,.\end{cases} \tag{6}$$

We will seek nontrivial solutions of Eq. (1) that satisfy the simplest boundary conditions

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$$u(a) = 0\,, \quad u'(b) = 0\,. \tag{7}$$

Here, it will be assumed that at the points of discontinuity of the coefficients the following internal conditions of conjugation are fulfilled:

$$[u]|_{x=x_i} = 0\,, \quad \left[p(x)\frac{du}{dx}\right]\Big|_{x=x_i} = 0\,. \tag{8}$$

Now we will introduce the new independent variable

$$z = \int_a^x \sqrt{\left(\frac{r(t)}{p(t)}\right)} dt = \sqrt{\left(\frac{r_1}{p_1}\right)}(x-a) +$$

$$+\sum_{i=1}^{n-1}\left(\sqrt{\left(\frac{r_{i+1}}{p_{i+1}}\right)} - \sqrt{\left(\frac{r_i}{p_i}\right)}\right)(x-x_i)\,S_-(x-x_i)\,. \tag{9}$$

Then differential equation (3) and boundary (7) and internal (8) conditions relative to the new variable will acquire, respectively, the form

$$\frac{d}{dz}\left[\sqrt{p(z)\,r(z)}\,\frac{du}{dz}\right] + \lambda^2\sqrt{p(z)\,r(z)}\,u = 0\,, \tag{10}$$

$$u(0) = 0\,; \quad u(z_*) = 0\,, \tag{11}$$

$$[u]|_{z=z_j} = 0\,; \quad \left[\sqrt{p(z)\,r(z)}\,\frac{du}{dz}\right]\Big|_{z=z_j} = 0\,, \tag{12}$$

where

$$\sqrt{p(z)\,r(z)} = \sqrt{p_1 z_1} + \sum_{j=1}^{n-1}\left(\sqrt{p_{j+1}r_{j+1}} - \sqrt{p_j r_j}\right) S_-(z-z_j)\,; \tag{13}$$

$$z_* = \int_a^b \sqrt{\left(\frac{r(t)}{p(t)}\right)} dt\,; \quad z_j = \int_0^{x_j} \sqrt{\left(\frac{r(t)}{p(t)}\right)} dt\,.$$

Next, we will reduce Eq. (10) to the partially degenerate equation

$$\frac{d^2u}{dz^2} + \lambda^2 u = -\sum_{j=1}^{n-1}\left(\sqrt{\left(\frac{p_{j+1}r_{j+1}}{p_j r_j}\right)} - 1\right)\frac{du}{dz}\Big|_{z=z_j}\,\delta_-(z-z_j)\,, \tag{14}$$

where $\delta_-(z-z_j)$ is the Dirac delta-function.

The general solution of Eq. (14) in mixed form is

$$u(z) = c_1 \sin\lambda z + c_2 \cos\lambda z - \frac{1}{\lambda}\sum_{j=1}^{n-1}\left(\sqrt{\left(\frac{p_{j+1}r_{j+1}}{p_j r_j}\right)} - 1\right)\frac{du}{dz}\Big|_{z=z_j} \times$$

$$\times \sin \lambda (z - z_j) S_-(z - z_j) . \tag{15}$$

The unknown derivatives in (15) are determined from the recursion relation

$$\sqrt{\left(\frac{p_{k+1} r_{k+1}}{p_k r_k}\right)} \frac{du}{dz}\bigg|_{z=z_k} = c_1 \lambda \cos \lambda z_k - c_2 \sin \lambda z_k -$$

$$- \sum_{j=1}^{k-1} \left(\sqrt{\left(\frac{p_{j+1} r_{j+1}}{p_j r_j}\right)} - 1 \right) \frac{du}{dz}\bigg|_{z=z_j} \cos \lambda (z_k - z_j) .$$

Without loss of generality in subsequent calculations, for simplicity we set $n = 2$. Then with account for

$$\frac{du}{dz}\bigg|_{z=z_1} = \sqrt{\left(\frac{p_1 r_1}{p_2 r_2}\right)} (c_1 \lambda \cos \lambda z_1 - c_2 \lambda \sin \lambda z_1) ,$$

the general solution of Eq. (15) can be represented in the following closed form:

$$u(z) = c_1 \left[\sin \lambda z - \left(1 - \sqrt{\left(\frac{p_1 r_1}{p_2 r_2}\right)} \right) \cos \lambda z_1 \sin \lambda (z - z_1) S_-(z - z_1) \right] +$$

$$+ c_2 \left[\cos \lambda z - \left(1 - \sqrt{\left(\frac{p_1 r_1}{p_2 r_2}\right)} \right) \sin \lambda z_1 \sin \lambda (z - z_1) S_-(z - z_1) \right] . \tag{16}$$

Let us analyze the obtained solution (16). Obviously, it is continuous in the closed interval $[0, z_*]$, i.e., the internal condition $[u]|_{z=z_1} = 0$ is fulfilled. At the same time we can show that the function $\sqrt{p(z) r(z)}\, u'(z)$ is continuous in the interval $[0, z_*]$. It is not difficult to verify this, since

$$\sqrt{p(z) r(z)} \frac{du}{dz} = c_1 \lambda_i \left\{ \sqrt{p_1 r_1} \cos \lambda z + \left(\sqrt{p_2 r_2} - \sqrt{p_1 r_1} \right) \times \right.$$

$$\left. \times [\cos \lambda z - \cos \lambda z_1 \cos \lambda (z - z_1)] S_-(z - z_1) \right\} -$$

$$- c_2 \lambda_i \left\{ \sqrt{p_1 r_1} \sin \lambda z + \left(\sqrt{p_2 r_2} - \sqrt{p_1 r_1} \right) \times \right.$$

$$\left. \times [\sin \lambda z - \sin \lambda z_1 \cos \lambda (z - z_1)] S_-(z - z_1) \right\}$$

and, consequently,

$$\sqrt{p_1 r_1} \frac{du}{dz}\bigg|_{z=z_1-0} = \sqrt{p_1 r_1} \frac{du}{dz}\bigg|_{z=z_1+0} ,$$

i.e., the second internal condition

$$\left[\sqrt{p(z) r(z)} \frac{du}{dz} \right]\bigg|_{z=z_1} = 0$$

is also fulfilled.

Having at hand the general solution (16) of Eq. (10), we can construct particular cases of it that satisfy the boundary conditions (11). Subjecting (16) to the first condition in (11), we find $c_2 = 0$. As a result, (16) acquires the form

$$u(z) = c_1 \left[\sin \lambda z - \left(1 - \sqrt{\left(\frac{p_1 r_1}{p_2 r_2} \right)} \right) \right] \cos \lambda z_1 \sin \lambda (z - z_1) S_-(z - z_1) . \tag{17}$$

Since

$$u'(z) = c_1 \lambda \left[\cos \lambda z - \left(1 - \sqrt{\left(\frac{p_1 r_1}{p_2 r_2} \right)} \right) \right] \cos \lambda z_1 \cos (z - z_1) S_-(z - z_1)$$

then for the second boundary condition in (11) to be fulfilled, it should be taken that

$$c_1 \lambda \left[\cos \lambda z_* - \left(1 - \sqrt{\left(\frac{p_1 r_1}{p_2 r_2} \right)} \right) \cos \lambda z_1 \cos \lambda (z_* - z_1) \right] = 0 . \tag{18}$$

We are interested only in nontrivial solutions of Eq. (18), and therefore $c_1 \neq 0$. Then from (18) it follows that

$$\cos \lambda z_* - \left(1 - \sqrt{\left(\frac{p_1 r_1}{p_2 r_2} \right)} \right) \cos \lambda z_1 \cos \lambda (z_* - z_1) = 0 ,$$

whence we find the characteristic equation for the eigenvalues $\lambda_n > 0$

$$\tan \lambda z_1 \tan \lambda (z_* - z_1) = \sqrt{\left(\frac{p_1 r_1}{p_2 r_2} \right)} \tag{19}$$

and the corresponding eigenfunctions

$$\varphi_n = \sin \lambda_n z - \left(1 - \sqrt{\left(\frac{p_1 r_1}{p_2 r_2} \right)} \right) \cos \lambda_n z_1 \sin \lambda_n (z - z_1) S_-(z - z_1) . \tag{20}$$

The orthogonality condition for the eigenfunctions can be obtained from Eq. (10) and conditions (11)-(12). For any pair of eigenfunctions $\varphi_i(z)$ and $\varphi_j(z)$ we have

$$\begin{aligned} &\frac{d}{dz} \left[\sqrt{p(z) r(z)} \frac{d\varphi_i}{dz} \right] + \lambda_i^2 \sqrt{p(z) r(z)} \varphi_i = 0 , \\ &\frac{d}{dz} \left[\sqrt{p(z) r(z)} \frac{d\varphi_j}{dz} \right] + \lambda_j^2 \sqrt{p(z) r(z)} \varphi_j = 0 . \end{aligned} \tag{21}$$

Multiplying the first equation by $\varphi_j(z)$ and the second equation by $\varphi_i(z)$, subtracting the second result from the first, and then integrating with respect to z in the interval from 0 to z_*, we arrive at

$$(\lambda_i^2 - \lambda_j^2) \int_0^{z_*} \sqrt{p(z) r(z)} \varphi_i(z) \varphi_j(z) dz = 0 , \quad i \neq j . \tag{22}$$

Hence it follows that eigenfunctions of (20) corresponding to different λ are orthogonal in the interval $[0, z_*]$ with weight $\sqrt{p(z) r(z)}$. Moreover, multiplying the first equation of (21) by $\varphi_j(z)$ and integrating with respect to z_*, with account for (11) and (22) we find that

$$\int_0^{z_*} \sqrt{p(z)\,r(z)}\,\varphi_i(z)\,\varphi_j(z)\,dz = \int_0^{z_*} \sqrt{p(z)\,r(z)}\,\varphi_i'(z)\,\varphi_j'(z)\,dz = 0\,,\ i\neq j\,, \tag{23}$$

i.e., the derivatives of eigenfunctions corresponding to different λ are also orthogonal with weight $\sqrt{p(z)r(z)}$.

Now we will normalize all eigenfunctions of the problem so as to fulfill the equality

$$\int_0^{z_*} \left[\frac{\varphi_i(z)}{N_i}\right]^2 \sqrt{p(z)\,r(z)}\,dz = 1\,, \tag{24}$$

whence for the normalization factors we obtain

$$N_i^2 = \int_0^{z_*} \sqrt{p(z)\,r(z)}\,[\varphi_i(z)]^2\,dz\,. \tag{25}$$

Calculating the normalization factors for functions (20) using formula (25), we find

$$N_i^2 = \frac{1}{2}\sqrt{p_1 r_1}\,z_1 + \frac{1}{2}\sqrt{p_2 r_2}\,(z_* - z_1)\left[1 - \left(1 - \frac{p_1 r_1}{p_2 r_2}\right)\cos^2\lambda_i z_1\right]. \tag{26}$$

Using the orthonormalized system of eigenfunctions of the boundary-value problem (10) and (11)

$$\Phi_i(z) = \frac{\varphi_i(z)}{N_i}\,, \tag{27}$$

we can represent an arbitrary function $f(z)$ prescribed in the interval $[0, z_*]$ in the form of the infinite series

$$f(z) = \sum_{k=1}^{\infty} A_k \Phi_k(z)\,, \tag{28}$$

where

$$A_k = \int_0^{z_*} \sqrt{p(z)\,r(z)}\,f(z)\,\Phi_k(z)\,dz = \frac{1}{N_k}\int_0^{z_*} \sqrt{p(z)\,r(z)}\,f(z)\,\varphi_k(z)\,dz\,. \tag{29}$$

As an example, we expand the following step function in a series in eigenfunctions (27) in the interval $[0, z_*]$:

$$f(z) = f_1 + (f_2 - f_1)\,S_-(z - z_1)\,.$$

Determining the coefficients of series (28) using formula (29), we obtain

$$A_k = \frac{f_1\sqrt{p_1 r_1}}{\lambda_k N_k}\left[1 + \left(\frac{f_2}{f_1} - 1\right)\cos\lambda_k z_1 - \frac{f_2}{f_1}\sqrt{\left(\frac{p_2 r_2}{p_1 r_1}\right)}\cos\lambda_k z_* + \right.$$
$$\left. + \frac{f_2}{f_1}\left(\sqrt{\left(\frac{p_2 r_2}{p_1 r_1}\right)} - 1\right)\cos\lambda_k z_1\cos\lambda_k(z_* - z_1)\right]. \tag{30}$$

Finally, we can represent the sought series in the form

$$f(z) = \sum_{k=1}^{\infty} \frac{f_1 \sqrt{p_1 r_1}}{\lambda_k N_k} \left[1 + \left(\frac{f_2}{f_1} - 1 \right) \cos \lambda_k z_1 - \frac{f_2}{f_1} \sqrt{\left(\frac{p_2 r_2}{p_1 r_1} \right)} \cos \lambda_k z_* + \right.$$

$$\left. + \frac{f_2}{f_1} \left(\sqrt{\left(\frac{p_2 r_2}{p_1 r_1} \right)} - 1 \right) \cos \lambda_k z_1 \cos \lambda_k (z_* - z_1) \right] \times$$

$$\times \left[\sin \lambda_k z_1 - \left(1 - \sqrt{\left(\frac{p_1 r_1}{p_2 r_2} \right)} \right) \cos \lambda_k z_1 \cos \lambda_k (z - z) S_- (z - z_1) \right]. \quad (31)$$

Upon going from the variable z to the previous variable x, the eigenfunctions (20) of the boundary-value problem (10)-(12) for $n = 2$ acquire the form

$$\varphi_n (x) = \sin \lambda_n \left[\sqrt{\left(\frac{r(x)}{p(x)} \right)} (x - x_1) + \sqrt{\left(\frac{r_1}{p_1} \right)} (x_1 - a) \right] +$$

$$+ \left(\sqrt{\left(\frac{p_1 r_1}{p_2 r_2} \right)} - 1 \right) \cos \lambda_n \sqrt{\left(\frac{r_1}{p_1} \right)} (x_1 - a) \sin \lambda_n \sqrt{\left(\frac{r(x)}{p(x)} \right)} (x - x_1) S_- (x - x_1) \quad (32)$$

and become orthogonal in the interval $[a, b]$ with weight $r(x)$. Here, the normalization factors can be determined using the formula

$$N_i^2 = \frac{1}{2} r_1 (x_1 - a) + \frac{1}{2} r_2 (b - x_1) \times$$

$$\times \left[\sin^2 \lambda_i \sqrt{\left(\frac{r_1}{p_1} \right)} (x_1 - a) + \cos^2 \lambda_i \sqrt{\left(\frac{r_1}{p_1} \right)} (x_1 - a) \right]. \quad (33)$$

Characteristic equation (19) for the eigenvalues λ_n is transformed to the form

$$\tan \lambda \sqrt{\left(\frac{r_1}{p_1} \right)} (x_1 - a) \tan \lambda \sqrt{\left(\frac{r_2}{p_2} \right)} (b - x_1) = \sqrt{\left(\frac{p_1 r_1}{p_2 r_2} \right)}. \quad (34)$$

Similarly, we can construct an orthonormalized system of eigenfunctions for a more general form of the boundary-value problem (1) and (2). It should be noted that since, in the variable z, boundary conditions of the fourth kind are fulfulled automatically, they do not need further consideration.

NOTATION

U, sought function; x, coordinate; α_k, β_k, prescribed positive numbers that do not vanish simultaneously; $p(x)$, $r(x)$, continuous functions in the closed interval $[a, b]$; $\delta_-(z-z_i)$, Dirac delta-function; c_1, c_2, integration constants,

REFERENCES

1. V. Ya. Arsenin, *Methods of Mathematical Physics and Special Functions* [in Russian], Moscow (1974).
2. E. M. Kartashov, *Analytical Methods in the Heat-Conduction Theory of Solids* [in Russian], Moscow (1985).
3. A. V. Luikov, *Heat-Conduction Theory* [in Russian], Moscow (1967).
4. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* [in Russian], Moscow (1972).